Dynamics of edge Majorana fermions in $\nu = \frac{5}{2}$ fractional quantum Hall effects

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Commencing with the composite fermion description of the $\nu = 5/2$ fractional quantum Hall effect, we study the dynamics of the edge neutral Majorana fermions. We confirm that these neutral modes are chiral and show that a conventional p-wave pairing interaction between CFs does not contribute to the dynamics of the edge neutral fermions. We find an important bilinear coupling between the charged and neutral modes. We show that owing to this coupling, the dispersion of the neutral modes is linear and their velocities are proportional to the wave vector of the charged mode. This dynamic origin of the motion of the edge Majorana fermions was never expected before.

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I. INTRODUCTION

Edge states in quantum Hall effects(QHE) play an extremely important role [1]. They are unique known media to reflect the topological order of the bulk states in fractional QHE (FQHE) [2]. Recently, the topological-protected quantum computation [3] based on the possible non-abelian statistics of quasiparticles in the filling factor $\nu = 5/2$ FQH (EFQH) states [4, 5, 6] has attracted great attentions for it provides a possible candidate for the decoherence-free qubit. Several experimental designs to detect the non-abelian statistics of the $\nu = 5/2$ EFQH state have been proposed [7]. The crucial part of these designs was the point-contact tunnelling of the quasiparticles between different edges of the EFQH droplet.

The fundamental physics behind the non-abelian statistics in the EFQH state is that low energy effective behaviors in the EFQH system are controlled by a $k = 2$ non-abelian Chern-Simons topological quantum field theory in the bulk while the behaviors of the edge states by a $c = 3k/(k+2) = 3/2$ conformal field theory which consists of chiral Majorana free fermionic modes with a velocity v_s and a chiral free bosonic mode with a velocity $v_n \ll v_s$. Although this was already recognized by Moore and Read [4] according to the correspondence between the wave functions of various FQH states and the correlation functions of the conformal field theory, a fundamental understanding from a microscopic point of view still lacks. The velocity v_n was not determined. The chirality of the edge modes is a mystery because the composite fermions (CFs) at the half-filling filling of the Landau level do not see the effective magnetic field.

The present author and his co-workers have provided an effective microscopic theory for the edge states of the odd denominator FQHE [8]. We used a mean field theory to the CF Hamiltonian in the bulk state. Project to a given Landau level, the electron band mass is renormalized to the CF effective mass determined by the Coulomb interaction. Shankar and Murthy have given an understanding for this matter from a Hamiltonian formalism [9]. A simplest calculation for the cancellation of the band mass has been given in our previous work from a Hartree-Fock approximation [10]. However, the random phase approximation calculation showed that the CF effective mass in the half-filled Landau level is logarithmically divergent no matter what gauge is taken [10, 11].

On the other hand, if there is a pairing interaction, this logarithmic divergence of the effective mass of vortex comes from the extended states with their energy larger than the paring gap [12]. The origin of divergence of the CF effective mass is similar. Therefore, the physics with the energy scale lower than the pairing gap can be studied in the same mean field approximation as that for the conventional FQHE if assuming a p-wave EFQH gap for the bulk CFs with the upmost Landau level being half filled. One can replace the band mass by the Hartree-Fock effective mass. For a pure p-wave superfluid, the pairing interaction gives a finite velocity of the edge Majorana fermionic excitation [13]. However, in this Letter, we will show that for the EFQH state, this conventional p-wave paring interaction does not contribute to the velocity of the edge Majorana fermionic modes. We assume the bulk states have a p-wave gap. By integrating out the bulk states, the effective theory of the edge states includes neutral chiral Majorana fermionic modes and a charged bosonic mode which is described by the Calogero-Sutherland model [14, 15]. Although a conventional p-wave pairing interaction between edge CFs does not contribute to the dynamics of the edge fermionic modes, there is a bilinear coupling between the neutral and charged modes. The velocity of the neutral modes is determined by this important coupling. It vanishes in the ground state as naively expected but linearly increases as the wave vector of the charged mode.

II. GENERAL DESCRIPTION

A two-dimensional interacting spinless electron gas in a high magnetic field is governed by the following Hamil-

tonian

$$H = \sum_{\alpha=1}^{N} \frac{1}{2m_b}[\vec{p}_\alpha - \frac{e}{c}\vec{A}(\vec{r}_\alpha)]^2 + \sum_{\alpha<\beta} V(\vec{r}_\alpha - \vec{r}_\beta) + \sum_{\alpha} U(\vec{r}_\alpha) + U_b, \quad (1)$$

where $V(\vec{r})$ is the interaction between electrons. m_b is the band mass of the electron; $U(\vec{r})$ is the external potential trapping the electron gas in a disc and U_b is the interacting potential of the neutralizing positive background charge [16]. The CF theory is a very useful tool in the FQHE physics [11, 17]. We begin with the CF transformation which reads $\Phi(z_1, ..., z_N) = \prod_{\alpha<\beta}\left[\frac{z_\alpha - z_\beta}{|z_\alpha - z_\beta|}\right]^{\tilde{\phi}} \Psi(z_1, ..., z_N)$, where Φ is the electron wave function and $\tilde{\phi}$ is an even integer. We assume the bulk states have a gap which is caused by a p-wave pairing of CFs for a filling factor $\nu = 1/\tilde{\phi}$ and all gapless excitations are in the edge. We now would like to study the effective theory of the CF edge excitations in a disc. The partition function of the system is given by

$$Z = \sum_{N^e} C_N^{N^e} \int_\partial d^2 z_1....d^2 z_{N^e} \int_B d^2 z_{N^e+1}...d^2 z_N \quad (2)$$
$$\times \left(\sum_\delta |\Psi_\delta|^2 e^{-\beta(E_\delta + E_g)} + \sum_\gamma |\Psi_\gamma|^2 e^{-\beta(E_\gamma + E_g)}\right),$$

where N_e is the CF number in the edge and N is the total electron number. We have divided the sample into the edge ∂ and the bulk B. E_g is the ground state energy and E_δ are the low-lying gapless excitation energies with δ being the excitation modes index. E_γ are the gapped excitation energies. One can integrate over the gapped state and the partition function of the system may be written as

$$Z \simeq \sum_{\delta, N^e} C_N^{N^e} \int_{\text{edge}} d^2 z_1...d^2 z_{N^e} |\Psi_{e,\delta}|^2 e^{-(\beta E_\delta(N^e) + E_g)} = \sum_{N^e} C_N^{N^e} \text{Tr}_{(\text{edge})} e^{-\beta H_e}. \quad (3)$$

In terms of the partition function (3), there is the most probable edge CF number $\bar{N}^e$ which is determined by $\delta Z/\delta N^e = 0$. $\bar{N}^e = \int dx \rho_e(x)$ with the edge density $\rho_e(x) = h(x)\bar{n}$. Here $h(x)$ is the edge deformation and $\bar{n}$ is the average density of the bulk electrons. We do not distinguish $\bar{N}_e$ and N_e hereafter if there is no ambiguity. We do not study the bulk physics in the present work and assume the CF interaction has been renormalized to a weak one. The electron band mass has been renormalized to m^*, the effective mass of the CF, which is finite with the order of the Coulomb interaction as we has explained. Hereafter, we use the unit $\hbar = e/c = 2m^* = 1$. For the disc sample with a radius R, the edge CFs are restricted in a circular strip near the boundary with its width $\delta R(\vec{r}) \ll R$. The edge Hamiltonian of CFs reads

$$H_e = \sum_{i=1}^{N^e} [\vec{p}_i - \vec{A}(\vec{r}_i)) + \vec{a}_e(\vec{r}_i) + \vec{a}_b(\vec{r}_i)]^2 + \sum_{i<j} V_{eff}(\vec{r}_i - \vec{r}_j) + \sum_i U_{eff}(\vec{r}_i), \quad (4)$$

where V_{eff} is the effective interaction between edge CFs and U_{eff} is the effective trapping potential including the interaction between the edge and bulk particles. The detailed expression of U_{eff} may be quantitatively important in numerical calculation [16], but here for simplicity, we suppose the trapping potential is an infinite wall for $r \geq R$ in order to analytically study a sharp edge state. Although this may not correctly reflect the quantitative behavior, the qualitative property which we are studying would not be changed. In real samples, the trapping potential is very dependent on the sample cleaving. If U_{eff} is not so sharp, the edge reconstruction is inevitable. In this case, more branches of the edge excitations may appear [18]. The statistics gauge field $\vec{a_e}$ is given by

$$\vec{a}_e(\vec{r}_i) + \vec{a}_b(\vec{r}_i) = \frac{\tilde{\phi}}{2\pi} \sum_{j\neq i} \frac{\hat{z} \times (\vec{r}_i - \vec{r}_j)}{|\vec{r}_i - \vec{r}_j|^2} + \frac{\tilde{\phi}}{2\pi} \sum_{\alpha \in \text{bulk}} \frac{\hat{z} \times (\vec{r}_i - \vec{r}_\alpha)}{|\vec{r}_i - \vec{r}_\alpha|^2}. \quad (5)$$

Taking the polar coordinate $z_i = r_i e^{i\varphi_i}$, the vector potential $A_\varphi(\vec{r}_i) = \frac{B}{2} r_i$ and $A_r(\vec{r}_i) = 0$ and substituting the polar variables and the vector potential to H_e, one has

$$H_e = \sum_i \Big[-\frac{\partial^2}{\partial r_i^2} + (-\frac{i}{r_i}\frac{\partial}{\partial \varphi_i} + \frac{\tilde{\phi}(N_e - 1)}{2r_i})^2 + \frac{\tilde{\phi}^2}{4R^2} \sum_i (\sum_{j\neq i} \cot\frac{\varphi_{ij}}{2})^2 - i\frac{\tilde{\phi}}{R} \sum_{i<j} \cot\frac{\varphi_{ij}}{2} \cdot [\frac{\partial}{\partial r_i} - \frac{\partial}{\partial r_j}] - \frac{1}{R}\frac{\partial}{\partial r_i} \Big] + V_{eff} + U_{eff} + O(\delta R/R), \quad (6)$$

where we use the mean field approximation $\bar{a}_{b,r} = 0$ and $\bar{a}_{b,r} = -\frac{B}{2} r_i$.

III. SOLUTIONS

Schrödinger equation $H_e \Psi = E\Psi$ holds for a FQH edge state with a general filling factor. For a given filling factor, the solution of Schrödinger equation is dependent on the bulk states as we have seen in solving the equations for the odd denominator FQHE [8]. For an even denominator quantum Hall state, e.g., the $\nu = 5/2$ state [19, 20], numerical calculations evidenced that Moore-Read Pfaffian state may be the ground state[21]. Motivated by the

bulk ground state, we can write the general form of the wave function as

$$\Psi(z_1,...,z_{N_e}) = \exp\left\{-i\sum_{i<j}\frac{r_i-r_j}{4R}\cot\frac{\varphi_{ij}}{2}\right\} \times\mathcal{A}(z_1,...,z_{N_e})f(r_1,...,r_{N_e})\phi_{cs}(\varphi_1,...,\varphi_{N_e}). \quad (7)$$

We anticipate that f determines the chirality of the edge charged mode which is described by ϕ_{cs}. Then, $\mathcal{A}$ corresponds to the neutral fermionic modes. In fact, Milovanovic and Read have written down the wave functions of the edge excitations [13, 22]. We will show that our solutions are consistent with their wave functions while we confirm that the neutral fermion modes indeed have a linear dispersion and are chiral. The new observation is that the velocity of this fermionic modes is proportional to the wave vector of the charged bosonic edge mode.

In terms of the general wave function, the problem ready to solve yields

$$\sum_i\left\{-\frac{\partial^2}{\partial r_i^2}-\frac{1}{R}\frac{\partial}{\partial r_i}-\frac{2}{R^2}\frac{\partial\ln\chi_{cs}}{\partial\varphi_i}\frac{\partial\ln\mathcal{A}}{\partial\varphi_i} -2\frac{\partial\ln\mathcal{A}}{\partial r_i}\frac{\partial}{\partial r_i}+U_{eff}\right\}f=(E-E_\varphi)f \quad (8)$$

with

$$-\sum_i\frac{1}{R^2}\frac{\partial^2}{\partial\varphi_i^2}\chi_{cs}+\frac{1}{4R^2}\sum_{i<j}\frac{\tilde{\phi}(\tilde{\phi}-1)}{\sin^2(\varphi_{ij}/2)}\chi_{cs}+V_{eff}\chi_{cs} = E_\varphi\chi_{cs}, \quad (9)$$

$$(\nabla^2\mathcal{A})\phi_{cs}=0,\quad \sum_i -i\frac{\partial\mathcal{A}}{\partial\varphi_i}=\sum_i l_i\mathcal{A}, \quad (10)$$

where $\chi_{cs}=e^{(i/2)\tilde{\phi}(N_e-1)\sum_i\varphi_i}\phi_{cs}$ and l_i is a set of integers or half-integers to be determined. Eq. (9) describes the charged edge excitations. If one neglects V_{eff} in Eq. (9), it is the famous Calogero-Sutherland model with $E_\varphi=\sum_i\frac{n_i^2}{R^2}$, whose solutions are $\chi_{cs}=\Phi(\mathbf{n})\prod_{i<j}[\sin\frac{\varphi_{ij}}{2}]^{\tilde{\phi}}$ where $\Phi(\mathbf{n})$ is the Jack polynomial [23] with the highest weight state satisfying

$$n_i=I_i+\frac{1}{2}\sum_{j\neq i}(\tilde{\phi}-1)\mathrm{sgn}(n_i-n_j), \quad (11)$$

where I_i are a set of integers with respect to physical momentum along the azimuthal direction. With the interaction V_{eff}, in the limit of dilute gas, $n_i=I_i+\frac{1}{2}\sum_{j\neq i}\theta(n_j-n_i)$ with $\theta(\Delta n)=(\tilde{\phi}-1)\mathrm{sgn}(\Delta n)-2\delta_{\tilde{\phi}-1}(\Delta n)$ [8]. The phase shift $2\delta_{\tilde{\phi}-1}(\Delta n)$ comes from the interaction V_{eff} which is continuous as a function of $\Delta k=\Delta n/R$ and vanishes at $\Delta k=0$ if V_{eff} is short range. Because of the step function $\mathrm{sgn}(\Delta n)$, the short range interaction may not affect the low-lying behavior of the edge modes. However, the velocity of the charged mode is lifted by the interaction [8, 24].(For details, see Refs. [24].) The Coulomb interaction may cause an additional charged branch of the edge excitations with a dispersion $\Delta k\ln\Delta k$ [8], which is precisely the one-dimensional plasmon excitation caused by the Coulomb interaction [25]. Now, we phenomenologically assume V_{eff} includes a conventional p-wave pairing interaction

$$H_{pair}=i\sum_i\Delta\cdot\begin{pmatrix}0 & \bar{\partial}_i\\ \partial_i & 0\end{pmatrix}, \quad (12)$$

where 2×2 matrix acts on a two-components of the complex spinless CF state [13]. For a pure p_x+ip_y superconductor without the bosonic mode, this p-wave paring interaction gives a gapless chiral fermion excitation with its velocity $v_n=\Delta$ at the edge [13]. However, for the EFQH state, the pairing interaction may only change the velocity of the charged mode but not contribute to that of the neutral modes since $\sum_i\partial\mathcal{A}/\partial z_i=\sum_i\partial\mathcal{A}/\partial\bar{z}_i=0$ according to the total antisymmetry of $\mathcal{A}$ with the exchange of the CFs.

Eqs.(10) imply that if we can find the eigen states of the second equation, which satisfy the first one, these states indeed describe the neutral Majorana fermion modes with a linear dispersion. The Moore-Read Pfaffian state $\mathrm{Pf}(\frac{1}{z_i-z_j})$ satisfies these equations with $\sum_i l_i=-2$. This is the ground state for even edge CF number. The ground state with odd edge CF is given by $\psi_l=\mathcal{A}\left(z_1^0\frac{1}{z_2-z_3}\cdots\frac{1}{z_{N_e-1}-z_{N_e}}\right)$. The multi Majorana fermion excited states are given by [13, 22]

$$\mathcal{A} = \mathcal{A}\left(z_1^{l_1}...z_s^{l_s}\frac{1}{z_{s+1}-z_{s+2}}...\right) = \mathcal{A}\left(\sum\epsilon_{i_1i_2...i_s}z_{i_1}^{l_1}z_{i_2}^{l_2}...z_{i_s}^{l_s}\psi_{i_1i_2...i_s}^{i_{s+1}...i_N}\right), \quad (13)$$

where $\psi_{i_1i_2...i_s}^{i_{s+1}...i_N}$ is a product of $\frac{1}{z_i-z_j}$ where z_i and z_j do not include $z_{i_1},z_{i_2},...,z_{i_s}$ and of course is symmetric for any permutation of $i_1,i_2,...,i_s$.

IV. THE DYNAMIC ORIGIN OF THE NEUTRAL FERMION VELOCITY AND CHIRALITY OF THE CHARGED MODE

We now study the dynamics of the neutral fermion and chirality of the edge excitations. As we have seen, the requirement of $(\nabla^2\mathcal{A})\phi_{cs}=0$ gives rise to the chirality of the neutral Majorana fermion edge modes. However, what is the origin of the motion of the neutral modes for V_{eff} does not contribute to v_n? On the other hand, the charged mode at the half-filling do not feel an effective magnetic field. The Calogero-Sutherland model has both left- and right-moving gapless modes from the Fermi points. Why is the edge charged mode still chiral? To answer these questions, we consider the radial equation (8). The pseudo-momentum of the Calogero-Sutherland model is defined by $k_i=n_i/R$. The pseudo-Fermi points are in

$\pm k_F = \pm\tilde{\phi}\bar{N}_e/2R$ and low energy excitations are around $k \sim \pm(k_F+q)$ for $q \ll k_F$. We define the neutral fermion momentum $p_i = l_i/R$ and consider multi Majorana fermion modes given by Eq.(13). Notice that $-i\frac{\partial \chi_{cs}}{\partial \varphi_i} = \sum_P k_{P_i}\chi_P$ and $-i\frac{\partial \mathcal{A}}{\partial \varphi_i} = \mathcal{A}\Big(\sum_{i_1 i_2 \ldots i_s}\epsilon_{i_1 i_2 \ldots i_s}(\delta_{i_1 i} l_1 + \ldots + \delta_{i_s i} l_s) z_{i_1}^{l_1} z_{i_2}^{l_2} \ldots z_{i_s}^{l_s}\psi_{i_1 i_2 \ldots i_s}^{i_{s+1}\ldots i_N}\Big)$, where P is a permutation of $\{1, \ldots N_e\}$. The fourth term in Eq. (8) reads

$$-\frac{2}{R^2}\sum_i \frac{\partial \ln \chi_{cs}}{\partial \phi_i}\frac{\partial \ln \mathcal{A}}{\partial \phi_i} = 2\sum_{a=1}^{s} p_a \sum_i k_i C_{k_i a} \quad (14)$$

where

$$\begin{aligned} C_{k_i a} &= \frac{1}{\chi_{cs}\mathcal{A}}\mathcal{A}\Big(\sum_{i_1,..,i_a,\ldots,i_s}\epsilon_{i_1\ldots i_a \ldots i_s} z_{i_1}^{l_1}\ldots z_{i_a}^{l_a}\ldots z_{i_s}^{l_s} \\ &\times \psi_{i_1 i_2 \ldots i_s}^{i_{s+1}\ldots i_N}\sum_{\{P|P_{i_a}=i\}}\chi_P\Big) \end{aligned} \quad (15)$$

The symmetry of the reflection gives that $C_{ka} = C_{-ka}$. Thus, we have $\sum_i k_i^{(0)} C_{k_i^{(0)} a} = 0$ for the ground state $\{k_i^{(0)}\}$. The low-lying excitations are given by $\{k_i^{(\pm)}\} = \{\pm(k_F+q), k_i^0 \neq \pm k_F\}$ for $q > 0$. Thus the radial equation reads

$$\begin{aligned} &\sum_i\Big\{-\frac{\partial^2}{\partial r_i^2} + 2k_F q \pm \sum_a 2qC_{\pm k_F a}p_a \\ &-2\Big(\frac{\partial \ln \mathcal{A}}{\partial r_i}\Big)\frac{\partial}{\partial r_i} + \frac{1}{R}\frac{\partial}{\partial r_i} + U_{eff}\Big\}f = E^{\pm}f. \end{aligned} \quad (16)$$

The third term in Eq. (16) is a bilinear coupling between the wave vectors of the neutral and charged modes. Since $C_{-k_F a} = C_{k_F a}$ which is assumed to be positive, this coupling means that the charged mode around k_F is accompanied by a set of neutral modes while the excitations around $-k_F$ is not physical because the accompanied neutral fermion modes lowered the energy of the system so that it becomes not low bounded. The physical excitations are confined around k_F. This proves the chirality of the charged edge bosonic mode with its velocity $v_s = v_F = 2k_F$. The neutral modes have velocities $v_{n,a} = 2C_{k_F a}q$. This is an important observation made in this work : $v_{n,a} \ll v_s$ is linearly dependent on the wave vector of the charged mode. This dynamic origin of the velocity of the edge Majorana fermion was never expected in the existed literature.

We emphasize that the origin of the chirality of the charged edge mode at the half-filling factor is very different from that for the FQHE in the odd denominator filling factor. For the conventional FQHE, the edge CFs see an effective magnetic field and the CFs' cyclotron motion in this effective magnetic field is the origin of the chirality. The CFs in a half-filling factor do not see such an effective magnetic field. The chirality of the charged bosonic edge mode stems from the coupling of this mode with the neutral Majorana chiral fermion modes. Of course, from the point of view of electron motion, the chirality of the edge excitations is still caused by the magnetic field. This is reflected in the wave functions: If we reverse the magnetic field, z should be replaced by $\bar{z}$ in all wave functions, which leads to a reverse of the chirality of the edge excitations.

V. CONCLUSIONS

We have studied the dynamics of the edge Majorana fermions for the EFQH state with $\nu = 5/2$. We found that there is a bilinear coupling between neutral and charged modes in the effective edge theory, which determines the dynamics of the neutral modes and the chirality of the charged modes. The velocity of the neutral fermion is proportional to the wave vector of the charged mode. The chirality of the charged edge mode origins from its coupling with the neutral modes.

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